A simple proof of the FWL Theorem

Here is a simple proof of the FWL Theorem (established in class). Before proceeding be sure that you understand the basic properties of M and X.

Start with:

$$Y = (P + M)Y = PY + MY = X_1\hat{\beta}_1 + X_2\hat{\beta}_2 + MY.$$

Multiply both sides by $X_1'M_2$ to get:

$$X_1'M_2Y = X_1'M_2X_1\hat{\beta}_1 + X_1'M_2X_2\hat{\beta}_2 + X_1'M_2MY$$

Now

- $M_2X_2 = 0$ by property of the M_2 matrix, so the second term of the right hand side vanishes.
- $X_1'M_2MY = X_1^{*'}e$, with $X_1^{*'} \equiv X_1'M_2$ and $e \equiv MY$.

  Now e are errors of the full regression of Y on X_1 and X_2. Note

  $$X'e = X'Mu = 0$$

  since $X'M = 0$. This implies that all columns of X are uncorrelated with e. $X_1^{*'}$ are residuals of regressing X_1 on X_2, that is, they are the part of X_1 not linearly explained by X_2. Consequently, by construction $X_1^{*'}$ is correlated with X_1 (though not with X_2). Then $X_1^{*'}e = 0$ since e is uncorrelated with $X_1^{*'}$. Hence the third term of the right hand side vanishes.

This leaves: $X_1'M_2Y = X_1'M_2X_1\hat{\beta}_1$.So:

$$\hat{\beta}_1 = (X_1'M_1X_1)^{-1}\, X_1'M_1Y = \tilde{\beta}_1$$

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